A small infinite geometric series problem

We need one formula for the following discussion:

$$S(\infty) = \frac{a}{1-r}, \quad -1 < r < 1,$$

where $S(\infty)$ is the sum of an infinite series with a, the first term and r, the common ratio.

Question Solve : $x^{-1} + x + x^2 + x^3 + \ldots = 3.5$, where $-1 < x < 1$.

(New Progress in Certificate Mathematics, Book 5A, Ex 1.3, No.36)

Method 1

Technique : ***Odd man out !*** x^{-1} is the "odd man" and should be taken away first.

$x^{-1} + x + x^2 + x^3 + \ldots = 3.5$

$x^{-1} + (x + x^2 + x^3 + \ldots) = 3.5$, the series inside the bracket is geometric.

$\frac{1}{x} + \frac{x}{1-x} = 3.5 \qquad \ldots.\ (1)$

For $x \neq 0, 1$,

$(1) \times x(1-x)$, $\quad (1 - x) + x^2 = 3.5x(1 - x)$

$2 - 2x + 2x^2 = 7x(1 - x)$

$2 - 2x + 2x^2 = 7x - 7x^2$

$9x^2 - 9x + 2 = 0$

$(3x - 2)(3x - 1) = 0$

$\therefore \quad x = \frac{2}{3} \quad \text{or} \quad x = \frac{1}{3}$.

Method 2

Technique : ***Fill the hole!*** "1" is the missing part and should be filled.

$x^{-1} + x + x^2 + x^3 + \ldots = 3.5$

$x^{-1} + 1 + x + x^2 + x^3 + \ldots = 3.5 + 1$

$\frac{1/x}{1-x} = \frac{9}{2}$

$2 = 9x(1 - x)$

$2 = 9x - 9x^2$

$9x^2 - 9x + 2 = 0$

$(3x - 2)(3x - 1) = 0$

$\therefore \quad x = \frac{2}{3} \quad \text{or} \quad x = \frac{1}{3}$.

Method 3

Technique : ***Remove the culprit!*** "The tail" of the infinite geometric series is the culprit.

$x^{-1} + x + x^2 + x^3 + \ldots = 3.5 \qquad \ldots.\ (1)$

$(1) \times x$, $\quad 1 + x^2 + x^3 + \ldots = 3.5x \qquad \ldots.\ (2)$

(1) – (2), $x^{-1} + x - 1 = 3.5 - 3.5x$ …. (3)

(3) × 2x, $2 + 2x^2 - 2x = 7x - 7x^2$

$9x^2 - 9x + 2 = 0$

$(3x - 2)(3x - 1) = 0$

$\therefore \quad x = \frac{2}{3}$ or $x = \frac{1}{3}$.

Method 4

Technique : ***Birds of the same feather flock together!*** Group the “like” terms.

$x^{-1} + x + x^2 + x^3 + \ldots = 3.5$

$(x^{-1} + x + x^3 + \ldots) + (x^2 + x^4 + \ldots) = 3.5$, there are two geometric series.

$$\frac{1/x}{1-x^2} + \frac{x^2}{1-x^2} = 3.5$$

$$\frac{1}{x} + x^2 = 3.5(1 - x^2)$$

$2 + 2x^3 = 7x - 7x^3$

$9x^3 - 7x + 2 = 0$

$(x + 1)(3x - 2)(3x - 1) = 0$, here you may need the factor theorem and rational zero th.

Since $x \neq -1$, $x = \frac{2}{3}$ or $x = \frac{1}{3}$.

Method 5

Technique : ***Simplify before acting!*** Simplification can remove the "fog" and let you see clearly.

$x^{-1} + x + x^2 + x^3 + \ldots = 3.5$ …. (1)

(1) × x, $1 + x^2 + x^3 + \ldots = 3.5x$, ***“simplify”***

Then you can treat this as original equation and use one of the above methods, for example:

$1 + x + x^2 + x^3 + \ldots = 3.5x + x$, “***fill the hole!***”

$$\frac{1}{1-x} = \frac{9x}{2}$$

$9x(1 - x) = 2$

$9x^2 - 9x + 2 = 0$

$(3x - 2)(3x - 1) = 0$

$\therefore \quad x = \frac{2}{3}$ or $x = \frac{1}{3}$.

Note : $x \neq 0$ in all the above discussion since it makes x^{-1} undefined.